A Necessary and Sufficient Condition for a Self-Diffeomorphism of a Smooth Manifold to be the Time-1 Map of the Flow of a Differential Equation

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- (A modified version of this might be used to find an isolated critical point of a differential equation on a Riemannian manifold M.)

Topological Dynamics

- Let M^n be a smooth manifold, let $h : M \to \mathbb{R}^n$ be a smooth map, and let $p \in M$ be an isolated zero of h.
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- (A modified version of this might be used to find an isolated critical point of a differential equation on a Riemannian manifold M.)
- If $Dh(p) : T_p(M) \to \mathbb{R}^n$ is nonsingular for all $p \in M$, define $f : M \to M$ by $f(p) = \exp\{Dh(p)^{-1}[-h(p)]\}$. Then f is a self-map of M of the kind that might be considered in smooth dynamics.

- One wants to know the behavior of the sequences $(x_i) = f^i(x_0)$ for various choices of x_0, in particular, when they converge to such an isolated zero p, when they tend towards a "cyclic sequence" of points $(p_1, p_2, \ldots, p_k)$, and when and how quickly they diverge to infinity.

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- Various sets, such as the Fatou set and the Julia set of f, can then be defined.

- Let M^n be a smooth manifold, let $\mathfrak{X}(M) = \{\zeta : M \to TM \mid \pi_M \circ \zeta = \mathrm{id}_M\}$ be the set of all the differential equations (also known as a tangent vector fields) on M, and let $\xi \in \mathfrak{X}(M)$.

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- (We are most interested in the case that $M = TQ$ for Q a closed, Riemannian manifold – the configuration space [or "c-space"] of a robot arm, for instance – and ξ is the vector field corresponding to an "elementary" Lagrangian on TQ [elementary with respect to a specific coordinate patch on Q]).

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- Then, if ξ is **complete**, ξ has a flow, $\Phi_t : M \times \mathbb{R} \to M$.
- Note 1) that ξ is always complete if M is closed, and 2) $\Phi_1 = f$ is a self-diffeomorphism of M.

- **Definition 1**

  Let M^n be a smooth manifold. Let $f : M \to M$ be a self-diffeomorphism of M. We say f is **rootable to the identity** if f is isotopic to the identity (note this implies f is orientation-preserving if M is orientable) and there is a sequence of self-diffeomorphisms (g_b) of M with

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- each g_b isotopic to the identity (note this implies each g_b is orientation-preserving if M is orientable)
- $(g_b)^b = f$ (that is, each g_b is a "b^{th} root of f")
- (**The commutativity condition**) for $b_1, b_2 \in \mathbb{N}$, $g_{b_1}[g_{b_2}(p)] = g_{b_1 b_2}(p) = g_{b_2 b_1}(p) = g_{b_2}[g_{b_1}(p)]$

Smooth Root Systems to the Identity

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- We call such a sequence (g_b) **a root system to the identity**. Note one should only need to determine (g_b) on some cofinal subset of the naturals that leads to a dense subset of the rationals, e.g., $b = 2^c$, leading to the dyadic rationals.

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- Set $\Phi_{\frac{a}{b}} : M \times \mathbb{Q} \to M$ by $\Phi_{\frac{a}{b}}(p) = (g_b)^a(p)$.

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- Set $\Phi_{\frac{a}{b}} : M \times \mathbb{Q} \to M$ by $\Phi_{\frac{a}{b}}(p) = (g_b)^a(p)$.
- We say f is **continuously rootable to the identity** if $\Phi_{\frac{a}{b}}$ extends to to a continuous function $\Phi_t : M \times \mathbb{R} \to M$ and call the root system to the identity (g_b) a **continuous root system to the identity** in this case.

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- Note that if such a continuous extension Φ_t exists, it is unique.

Smooth Root Systems to the Identity

- We say f is **smoothly rootable to the identity** if Φ_t is smooth and call the root system to the identity (g_b) a **smooth root system to the identity** in this case.

Main Result

- **Theorem 1 (A Necessary and Sufficient Condition for a Self-Diffeomorphism of a Manifold to be the Time-1 Map of the Flow of a Differential Equation)**

  Let $n \in \mathbb{Z}^+$, M^n be a closed, smooth manifold, and f be a self-diffeomorphism of M. Then $f = \Phi_1$ for Φ_t the flow of a complete differential equation on M if and only if f is smoothly rootable to the identity. In this case, for a smooth root system to the identity (g_b) and corresponding flow Ψ_t, the differential equation ξ inducing Ψ_t is given by
  $\xi = \frac{\partial}{\partial t} \{\Psi_t\} \Big|^{t=0}$.

Examples

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- Note that the self-diffeomorphism P of S^1 defined by $P(p) = \overline{p}$ (complex conjugation, picturing S^1 as being the unit circle in $\mathbb{C}$) has $DP^{-1} \circ \xi_2 \circ P = \xi_1$,for ξ_i the differential equation determined by $(g_{i,b})$, where $DP : TS^1 \to TS^1$ is the induced map on the tangent bundle.

Examples

- **Example 2**

  (A generalization of an example from Jason Devito) With S^3, thinking of S^3 as a Lie group, the antipodal map (left multiplication by -1, L_{-1}) has uncountably many square roots: left multiplication by any purely imaginary unit quaternion.

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- Every imaginary quaternion has exactly two quaternion square roots, u_3 and $-u_3$ with $(\pm u_3)^2 = q$. Only one of the $\pm u_3$'s at level 3 will have a angle smaller that q with 1, the other one will be $-u_3$ and will have a smaller angle than q with -1.

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- This pattern continues with u_{c-1} has exactly two quaternion square roots, u_c and $-u_c$ with $(\pm u_c)^2 = u_{c-1}$. Only one of the $\pm u_c$'s at level c will have a angle smaller that u_{c-1} with 1, the other one will be $-u_c$ and will have a smaller angle than $-u_{c-1}$ with -1.

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- If $g_{2^c} = L_{u_c}$, then (g_{2^c}) is a sequence of 2^c th roots of $f = L_{-1}$ defined on a cofinal subset of the naturals with each (g_{2^c}) a smooth root system to the identity.

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- If $g_{2^c} = L_{u_c}$, then (g_{2^c}) is a sequence of $2^{c\ \text{th}}$ roots of $f = L_{-1}$ defined on a cofinal subset of the naturals with each (g_{2^c}) a smooth root system to the identity.
- Hence, we have a case where we have uncountably many different differential equations ξ_q with $\Phi_{q,t=1} = f$.

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- Hence, we have a case where we have uncountably many different differential equations ξ_q with $\Phi_{q,t=1} = f$.
- Note that if q_1 and q_2 are distinct imaginary quaternions with $q_1^2 = q_2^2 = -1$, the the self-diffeomorphism P of S^3 defined by $P(p) = L_{q_2(q_1^{-1})}$has $DP^{-1} \circ \xi_2 \circ P = \xi_1$,for ξ_i the differential equation determined by $(g_{i,b})$, where $DP : TS^3 \to TS^3$ is the induced map on the tangent bundle.

Proof of the Main Result

- **Lemma 1**

  Let (g_b) be a root sequence to the identity, let $a_1, a_2 \in \mathbb{Z}$, and let $b_1, b_2 \in \mathbb{Z}^+$. Then $g_{b_2}^{a_2}[g_{b_1}^{a_1}(p)] = (g_{b_1 b_2})^{a_1 b_2 + a_1 b_2}(p)]$

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- **Proof.**

  $g_{b_2}^{a_2}[g_{b_1}^{a}(p)] = g_{b_1 b_2}^{a_2 b_1}[g_{b_1 b_2}^{a_1 b_2}(p)]$ by the coherency condition

  $g_{b_2}^{a_2}[g_{b_1}^{a}(p)] = (g_{b_1 b_2})^{a_2 b_1 + a_1 b_2}(p)$ by the definition of composition of functions. ■

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  ($\Rightarrow$) Suppose $f = \Phi_1$ for Φ_t the flow of a differential equation ξ on M. Define $H : M \times \mathbb{I} \to M$ by $H(p,t) = \Phi(p, 1-t)$.

- **Proof.**

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- Then $H(p, 0) = \Phi_{1-0}(p) = \Phi_1(p) = f(p)$ and $H(p, 1) = \Phi_{1-1}(p) = \Phi_0(p) = \mathrm{id}_M(p)$, so f is isotopic to id_M.

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- Let $b \in \mathbb{Z}^+$ and set $g_b(p) = \Phi_{\frac{1}{b}}(p)$.

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- Let $b \in \mathbb{Z}^+$ and set $g_b(p) = \Phi_{\frac{1}{b}}(p)$.
- Then each g_b is isotopic to id_M and $(g_b)^b(p) = \Phi_{\frac{1}{b}}^b(p) = \Phi_{b\frac{1}{b}}(p) = \Phi_1(p) = f(p)$, so each g_b is a "b^{th} root of f".

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- Let $b_1, b_2 \in \mathbb{Z}^+$. Then

$$\begin{aligned} g_{b_1}[(g_{b_2})(p)] &= \Phi_{\frac{1}{b_1}}[\Phi_{\frac{1}{b_2}}(p)] \\ &= \Phi_{\frac{1}{b_1}+\frac{1}{b_2}}(p) \text{ as } \Phi_t \text{ is a flow} \\ &= \Phi_{\frac{1}{b_2}+\frac{1}{b_1}}(p) \\ &= \Phi_{\frac{1}{b_2}}[\Phi_{\frac{1}{b_1}}(p)] \\ &= g_{b_2}[(g_{b_1})(p)] \end{aligned},$$

so (g_b) obeys the commutativity condition.

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so (g_b) obeys the commutativity condition.
- Next, for $a \in \mathbb{Z}$,
$$\begin{aligned} (g_b)^a(p) = (\Phi_{\frac{1}{b}})^a(p) &= \Phi_{\frac{a}{b}}(p) \\ &= \Phi_{\frac{\frac{a}{GCD(a,b)}}{\frac{b}{GCD(a,b)}}}(p) \text{ as } \Phi_t \text{ is a flow} \\ &= \left(g_{\frac{b}{GCD(a,b)}}\right)^{\frac{a}{GCD(a,b)}}(p) \end{aligned}$$
so (g_b) obeys the coherency condition.

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- Finally, $\lim_{b\to\infty}(g_b)(p) = \lim_{b\to\infty}\Phi_{\frac{1}{b}}(p) = \Phi_0(p) = \mathrm{id}_M$. $\blacksquare(\Rightarrow)$

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- $(\Leftarrow)$ Suppose f is smoothly rootable to the identity. Let (g_b) be a smooth root system to the identity for f.

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- $(\Leftarrow)$ Suppose f is smoothly rootable to the identity. Let (g_b) be a smooth root system to the identity for f.
- Define $\Psi_{\frac{a}{b}} : M \times \mathbb{Q} \to M$ by $\Psi_{\frac{a}{b}}(p) = (g_b)^a(p)$.

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- Then

$$\begin{aligned}
\Psi_{\frac{a}{b}}(p) &= (g_b)^a(p) \\
&= \left(g_{\frac{b}{GCD(a,b)}}\right)^{\frac{a}{GCD(a,b)}}(p) \\
\Psi_{\frac{a}{b}}(p) &= \Psi_{\frac{\frac{a}{GCD(a,b)}}{\frac{b}{GCD(a,b)}}}(p)
\end{aligned}$$

by the coherency condition, so $\Psi_{\frac{a}{b}}$ is well-defined.

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- $(\Leftarrow)$ Suppose f is smoothly rootable to the identity. Let (g_b) be a smooth root system to the identity for f.
- Define $\Psi_{\frac{a}{b}} : M \times \mathbb{Q} \to M$ by $\Psi_{\frac{a}{b}}(p) = (g_b)^a(p)$.
- Then
$$\begin{aligned} \Psi_{\frac{a}{b}}(p) &= (g_b)^a(p) \\ &= \left(g_{\frac{b}{GCD(a,b)}}\right)^{\frac{a}{GCD(a,b)}}(p) \\ \Psi_{\frac{a}{b}}(p) &= \Psi_{\frac{\frac{a}{GCD(a,b)}}{\frac{b}{GCD(a,b)}}}(p) \end{aligned}$$
by the coherency condition, so $\Psi_{\frac{a}{b}}$ is well-defined.
- Moreover, as (g_b) is a smooth root system to the identity, it is a continuous root system to the identity, so $\Psi_{\frac{a}{b}}$ extends uniquely to a continuous function $\Psi_t : M \times \mathbb{R} \to M$.

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- Note $\Psi_0 = \Psi_{\frac{0}{b}} = (g_b)^0 = \mathrm{id}_M$.
- Let $a_1, a_2 \in \mathbb{Z}$ and $b_1, b_2 \in \mathbb{Z}^+$. Then

$$\begin{aligned}
\Psi_{\frac{a_2}{b_2}}\left[\Psi_{\frac{a_1}{b_1}}(p)\right] &= (g_{b_1})^{a_1}[(g_{b_2})^{a_2}(p)] \\
&= (g_{b_1 b_2})^{a_2 b_1 + a_1 b_2}(p) \text{ by Lemma 1} \\
&= \Psi_{\frac{a_2 b_1 + a_1 b_2}{b_1 b_2}}(p) \\
&= \Psi_{\frac{a_2 b_1}{b_1 b_2} + \frac{a_1 b_2}{b_1 b_2}}(p) \\
\Psi_{\frac{a_2}{b_2}}\left[\Psi_{\frac{a_1}{b_1}}(p)\right] &= \Psi_{\frac{a_2}{b_2} + \frac{a_1}{b_1}}(p) \text{ by the fact that } \Psi_{\frac{a}{b}} \\
&\quad \text{is well-defined on } \mathbb{Q}.
\end{aligned}$$

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- It then follows that $\Psi_{t_2}[\Psi_{t_1}(p)] = \Psi_{t_2+t_1}(p)$ for $t_1, t_2 \in \mathbb{R}$ by continuity as (g_b) is a continuous root system to the identity.

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- A proof of the fact that for a complete, time-independent vector field ξ, the differential equation ξ inducing Φ_t is given by $\xi = \frac{\partial}{\partial t}\{\Phi_t\}\Big|^{t=0}$ may be found in, for instance, [1]. $\blacksquare$

References

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The End

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