More Examples of Pseudo-Collars on High-Dimensional Manifolds

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Quillen's Plus Construction

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- If the object is a CW complex, the Plus Construction simply create a map f between the two CW complexes
- In either case, the map f is a $\mathbb{Z}Q$-homology isomorphism

Semi-S-Cobordisms

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Theorem 1 (R., 2015)

Let M be a manifold of dimension 6 or higher, with $\pi_1(M) \cong Q$. Let K be a finitely presented superperfect group. Let G be a semi-direct product of Q by K, $G = K \rtimes Q$. Then there is a cobordism (W, M, M_-) with $\pi_1(M_-) \cong G$ and $M \hookrightarrow W$ a simple homotopy equivalence

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- (Note that G being a semi-direct product of Q by K, $G = K \rtimes Q$, means G satisfies $1 \to K \to G \to Q \to 1$, so G is a group extension of Q by K, with a special condition for how elements of Q multiply elements of K in G)

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- (Semi-direct products are the simplest kind of group extensions; direct products are one example)

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- Note (W, M_-, M) (read upside-down, with the roles of M and M_- reversed) is a plus cobordism (so $(M_-)^+ \approx M$)

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- Gluing W_1 and W_2 together across M_- and so on produces an $(n+1)$-dimensional, 1-ended manifold V whose neighborhoods of infinity are pseudo-collared

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- This is one advantage of using semi-direct products over direct products

Uncountable Many Pseudo-Collars

- A subset A of a finitely presented group S is called an unpermutable set if is countably infinite and has the property that if $\phi : K \to K$ is an isomorphism and $\phi(a_i) = a_j^{\pm 1}$, then $i = j$

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Theorem 2 (R., 2020)

Let M be a manifold of dimension $n \geq 6$ with fundamental group $\mathbb{Z}$. Let S be a finitely presented, superperfect, centerless, freely indecomposable, Hopfian, co-Hopfian group with an unpermutable set, and let $K = S * S$. Then there are uncountably many $(n+1)$-dimensional, pseudo-collarable, 1-ended manifolds V with boundary M

- V will break up into semi-s-cobordisms (W_j, M_{j-1}, M_j), where $G_j \cong S \rtimes G_{j-1}$, $G_j = \pi_1(M_j)$

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- We will produce different G_j's by varying the outer actions, a technical part of semi-direct products, while keeping the quotient group, $\mathbb{Z}$, and kernel group, S, essentially constant
- We will produce one V for each $\omega \in \Pi_{i=1}^{\infty}\{0, 1\}$

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- Of course, it's pretty easy to whip out a 1-ended, pseudo-collarable manifold V: you just keep taking larger and larger G_j's and M_j's and glue the cobordisms together
- The hard part is proving that the resulting pro-fundamental group systems at infinity are all non-isomorphic
- For example, if $Q = \Pi_{i=1}^{\infty}\mathbb{Z}$, $K_1 = \mathbb{Z}$, and $K_2 = \mathbb{Z} \times \mathbb{Z}$, then $G_1 = K_1 \times Q$ and $G_2 = K_2 \times Q$ are isomorphic, even though $K_1 \ncong K_2$

Sketch of Proof

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Lemma 3

Let A, B, C, and D be nontrivial groups and ley $\phi : A \times B \to C * D$ be a epimorphism. Then either $\phi(A \times \{1\})$ is all of $C * D$ and $\phi(\{1\} \times B)$ is trivial or $\phi(A \times \{1\})$ is trivial and $\phi(\{1\} \times B)$ is all of $C * D$

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- Remark This is really the grain of sand that led to the pearl that is the theorem. Everything must commute and the domain, and nothing can commute in the range

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- The proof uses the fact that a free product is never an internal direct product

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Lemma 4 (The Straightening-Up Lemma ($n = m$))

Let $n = m$, let K be a free product, and let $\psi : K \times K \times \ldots \times K$ (n copies) $\to K \times K \times \ldots \times K$ (m copies) be an isomorphism. Write $\psi_{i,j}$ for $\pi_{K_j} \circ \psi|_{K_i}$. Then ψ splits as n isomorphisms $\psi_{i,\sigma(i)}$, with σ a permutation, with all other $\psi_{i,j}$'s being the trivial map

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Lemma 5 (The Straightening-Up Corollary ($n > m$))

Let $n > m$, let K be a free product of Hopfian groups, and let $\psi : K \times K \times \ldots \times K$ (n copies) $\rightarrow K \times K \times \ldots \times K$ (m copies) be an epimorphism. Write $\psi_{i,j}$ for $\pi_{K_j} \circ \psi|_{K_i}$. Then ψ splits as m isomorphisms $\psi_{\sigma^{-1}(i),i}$, with σ a permutation, with all other $\psi_{i,j}$'s being the trivial map

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- where ϕ_{u_i} is the outer action of $\mathbb{Z}$ on K_i given by
$$\phi_{u_i}(z)(p) = \begin{cases} p & \text{if } p \in S_1 \\ u_i^{-z} p u_i^{z} & \text{if } p \in S_2 \end{cases}$$

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- (This particular kind of outer action is called a partial conjugation)

Lemma 6 (The Conder Isomorphism Lemma ($n = m$))

Let $n = m$, and let $\theta : G_1 \to G_2$ be an isomorphism. Then θ restricts to an isomorphism on the commutator subgroup $C = K \times K \times \ldots \times K$, and the K factors which correspond by the Straightening-Up Lemma ($n = m$) have ϕ_{u_i}'s being determined by the same u_i in the definition of the unpermutable group A

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Lemma 7 (The Conder Isomorphism Corollary ($n > m$))

Let $n > m$, and let $\theta : G_1 \to G_2$ be an epimorphism. Then θ restricts to an epimorphism on the commutator subgroup $C = K \times K \times \ldots \times K$, and the K factors which correspond by the Straightening-Up Corollary ($n > m$) have ϕ_{u_i}'s being determined by the same u_i in the definition of the unpermutable group A

Sketch of Proof

- ($\Rightarrow$) Suppose there is an isomorphism θ between the two extensions. Then θ must preserve the commutator subgroup, a characteristic group, so it induces an automorphism of $K_1 \times \ldots \times K_n$, say ψ. By Lemma 4, θ must send each of the n factors of $K_1 \times \ldots \times K_n$ in the domain isomorphically onto exactly one of the n factors of $K_1 \times \ldots \times K_n$ in the range. Let σ be the permutation from Lemma 4.

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- Also, the associated map to θ on quotient groups must send the infinite cyclic quotient $G_{(k_1,\ldots,k_n)}/(K_1 \times \ldots \times K_n)$ isomorphically onto the infinite cyclic quotient $G_{(l_i,\ldots,l_n)}/(K_1 \times \ldots \times K_n)$. So, θ takes the generator, z, of $\mathbb{Z}$ in $G_{(k_1,\ldots,k_n)}$ to the an element cw^e in $G_{(l_i,\ldots,l_n)}$, where c is some element of $K_1 \times \ldots \times K_n$, w generates the $\mathbb{Z}$ in $G_{(l_1,\ldots,l_n)}$, and e is $+1$ or -1.

Sketch of Proof

- But also we know that z centralises the factor $S_{i,1}$ in K_i, so its θ-image cw^e must centralize $\theta(S_{i,1}) = \psi(S_{i,1}) = Q_i$, say, in $K_{\sigma(i)}$, the copy of K to which K_i is sent under the isomorphism given by Lemma 4, and act as conjugation by $l_{\sigma(i)}$ on $\theta(S_2) = \psi(S_2) = R_i$, say, in $K_{\sigma(i)}$.

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- By the Kurosh Subgroup Theorem, $Q_i = F * B_1^{d_1} * B_2^{d_2} * \ldots * B_j^{d_j}$, where F is free, each $d \in K_{\sigma(i)}$ and $B_j \leq S_{\sigma(i),u_j}$, $u_j \in \{1,2\}$. Since the Abelianization of Q is trivial, we have that F is trivial. As $Q_i \cong S$ and S is freely indecomposable, we must have that $j = 1$. As S is co-Hopfian, we must have that $B_1 = S_{\sigma(i),u_j}$. Now, this implies that conjugation by c has the same effect as conjugation by w^{-e} on the subgroups Q_i of $K_{\sigma(i)}$, which is isomorphic to $S_{i,1}$, and Q_j of $K_{\sigma(j)}$, which is isomorphic to $S_{j,1}$, for $i \neq j$.

Sketch of Proof

- This implies $Q_i^c = Q_i^{w^{-e}}$ pointwise and $Q_j^c = Q_j^{w^{-e}}$ pointwise; as this would require $c \in K_{\sigma i} \cap K_{\sigma j}$ for $i \neq j$, we conclude $c = e$. Thus θ takes z to w^e.

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- So, $Q_i = \theta(S_{i,1}) = S^d_{\sigma(i),k_j}$, with $d \in K_{\sigma(i)}$ and $k_1 \in \{1, 2\}$.

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- So, $Q_i = \theta(S_{i,1}) = S^d_{\sigma(i),k_j}$, with $d \in K_{\sigma(i)}$ and $k_1 \in \{1, 2\}$.
- By a symmetric argument, $R = E * C_1^{f_1} * C_2^{f_2} * \ldots * C_j^{f_j}$, where E is free, each $f \in S_{\sigma(i)}$ and $C_i \leq S_{\sigma(i),v_i}$, $v_i \in \{1, 2\}$ and $R = \theta(S_{i,2}) = S^f_{\sigma(i),l_i}$, with $f \in K_{\sigma(i)}$ and $l_i \in \{1, 2\}$.

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- Since θ is onto, the restriction of θ to K_i must be onto. If $k_i = l_i$, then the projection onto the $S_{i,r}$ factor of $\theta\big|_{K_i}$, where $r \in \{1, 2\}\backslash\{k_i\}$, would be trivial, and $\theta\big|_{K_i}$ would not be onto. Thus, $l_i \in \{1, 2\}\backslash\{k_i\}$.

Sketch of Proof

- Suppose for some K_i, $\theta(S_{i,1}) = S^d_{\sigma(i),2}$ and $\theta(S_{i,2}) = S^f_{\sigma(i),1}$. Let $x \in S_{i,2}$ be an element other than the identity element and let $y = \theta(x) \in Q$. Then

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- $\theta(x^z) = \theta(x)^{\theta(z)}$

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- $\theta(x^z) = \theta(x)^{\theta(z)}$
- $\theta(x^z) = y^{\theta(z)}$
  $\theta(x^{u_i}) = y$ as conjugation by $\theta(z)$ must act trivially on Q
  $\theta(x^{u_i}) = \theta(x)$, which shows θ is not 1-1 for all $u_i \in A$ with u_i not the identity, which contradicts the fact that θ is an isomorphism.

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  $\theta(x^{u_i}) = \theta(x)$, which shows θ is not 1-1 for all $u_i \in A$ with u_i not the identity, which contradicts the fact that θ is an isomorphism.
- It follows that $\theta(S_1) = S_1^d$ and $\theta(S_2) = S_2^f$

Sketch of Proof

- Now, suppose $u_i, v_{\sigma(i)} \in A$ and $\theta(u_i) \neq v_{\sigma(i)}$, say $\theta(u_i) = q$. If x is any element of the $S_{i,2}$ factor in K_i other than the identity, set $y = \theta(x) = \psi(x)$. Then

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- By the uniqueness of normal forms of elements in free products, $q = v_{\sigma(i)}^e$

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- Finally, if $u_i, v_{\sigma(i)} \in A$ and x is any element of the $S_{i,2}$ factor in K_i other than the identity, set $y = \theta(x) = \psi(x)$. Then

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- But now, $\psi(u_i) = (v_{\sigma(i)})^e$; since no isomorphism of $S_{i,1}$ takes any given element of A onto any other element of A or the inverse of any other element of A, we must have $\theta(u_i) = u_i^{\pm 1}$. QED Lemma

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- In my disseratation, we had one group, Thompson's group V, that met all the requirements for a kernel group and had torsion elements of all orders.
- Already, due to a suggestion by Jason Manning, we have a countable collection of fundamental groups of hyperbolic 3-manifolds that are allowable and meet all the requirements for a kernel group, for each of which we produce an uncountable collection of pseudo-collars.

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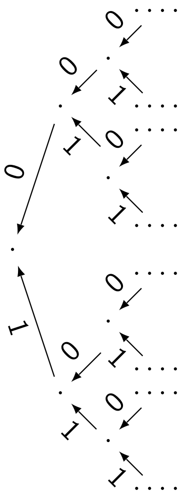

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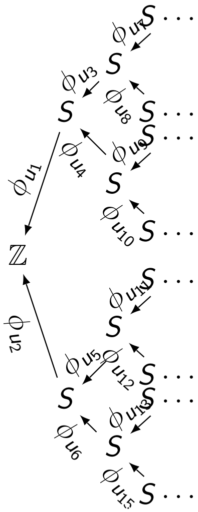

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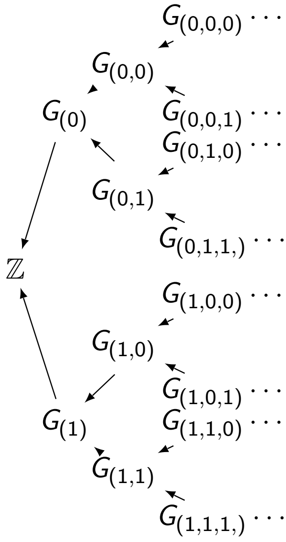

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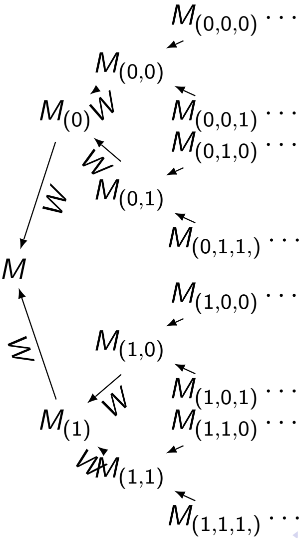

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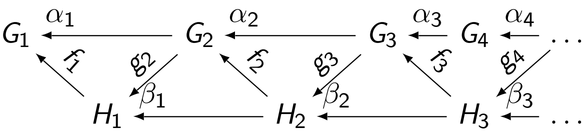
- Corresponding to each $\omega \in \Omega$, we have an inverse sequence of groups $(G_{(\omega,n)}, \alpha_{(\omega,n)})$
- Two inverse sequences are pro-isomorphic if and only if, after passing to subsequences, they may be put into a ladder diagram



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- Suppose ω and η agree up to some level n_0
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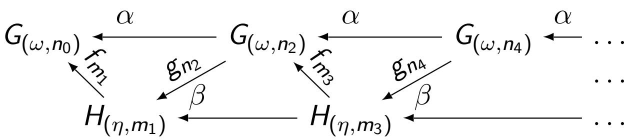
- Suppose, after passing to subsequences, we have their pro-fundamental group systems at infinity fitting into a ladder diagram



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- This concludes the proof

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