Some Results on Pseudo-Collar Structures on High-Dimensional Manifolds

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- In either case, the map f is a $\mathbb{Z}Q$-homology isomorphism

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- (Semi-direct products are the simplest kind of group extensions; They are the split-exact extensions. Direct products are one example)
- The cobordism (W, N, N_-) is has $\pi_1(N) \cong Q$, $\pi_1(N_-) \cong G$, and $N \hookrightarrow W$ a simple homotopy equivalence

1-Sided s-Cobordisms

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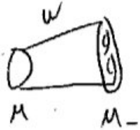

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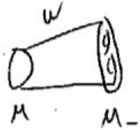

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- Note (W, N_-, N) (read upside-down, with the roles of N and N_- reversed) is also a 1-sided s-cobordims, called a plus cobordism (so $(N_-)^+ \approx N$)

Geometric Reverse to the Plus Construction

- For the rest of this talk, all mflds are assumed to be orientable.

Theorem (R., 2009)

Let $n \geq 6$. Let N be a closed n-manifold with $Q = \pi_1(N)$. Let S be f.p. and superperfect. Then there exists a cpt n-dimensional 1-sided h-cobordism (W, N, M) with left-hand bdy N and with right-hand bdy M having $\pi_1(M) \cong G$, where G is a gp extension of Q by S with trivial second homology elt and with $N \hookrightarrow W$ a simple homotopy equivalence.

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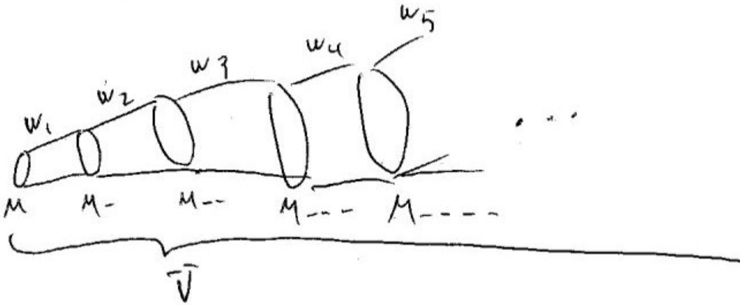
- 

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- This is one advantage of using semi-direct products over direct products

Uncountable Many Pseudo-Collars

Theorem (R., 2014)

Let M be a manifold of dimension $n \geq 6$ with fundamental group $\mathbb{Z}$. Let P be a finitely presented, superperfect, centerless, Hopfian group with torsion elements of infinitely many different orders, and let $S = P * P$. Then there are uncountably many $(n+1)$-dimensional, pseudo-collarable, 1-ended manifolds V with boundary M

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- We will produce one V for each increasing sequence of prime numbers

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Theorem (R., 2010)

Let M^n be an orientable, closed manifold ($n \geq 6$) such that $\pi_1(M)$ contains an element t_0 of infinite order and $\pi_1(M)$ is hypo-Abelian. Then there exists a 1-ended, orientable manifold V^{n+1} with $\partial V = M$ in which all clean neighborhoods of infinity have finite homotopy type, but which does not have perfectly semistable fundamental group at infinity. Thus, V^{n+1} is absolutely inward tame but not pseudocollable.

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- A group G is hypo-Abelian if its perfect core, the largest perfect subgroup (necessarily normal) $\langle e \rangle$.

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- However, we supply a new proof of this fact.

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- Take a small disk D inside of $N \times \{1\}$.
- Attach trivial 1-handles inside the disk for each generator β of P.
- Attach 2-handles across the disk for each relator s of P.

- As a corollary to the Solution to the Group Extension Problem, since we are using the trivial 2nd homology element, for each element $\beta_i\alpha_j$ in G, there is a word $w_{i,j}$ in the β's so that $\beta_i\alpha_j = \alpha_j w_{i,j}$.

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- Attach a 2-handle across each $\beta_i \alpha_j (\alpha_j w_{i,j})^{-1}$ (these will leak outside the disk).
- Now, we have a cobordism (W_1, N, M) with$\pi_1(N) \cong Q$, $\pi_1(W_1) \cong G$, and $\pi_1(M) \cong G$.

Lemma

Let $\overline{W_1}$ be the cover of W_1 with fund gp P (and covering transformation group Q). Then we may arrange the handles attached to $\overline{W_1}$ across $\overline{M}$ so that they project down equivariantly via the covering map to corresponding handles attached to W across M.

- So, now the relative handlebody chain complex of $(\overline{W_1}, \widetilde{N})$ looks like

$$\begin{array}{ccccccccc} \to C_3(\overline{W_1}, \widetilde{N}; \mathbb{Z}) & \longrightarrow & C_2(\overline{W_1}, \widetilde{N}; \mathbb{Z}) & \xrightarrow{\partial} & C_1(\overline{W_1}, \widetilde{N}; \mathbb{Z}) & \longrightarrow & C_0(\overline{W_1}, \widetilde{N}; \mathbb{Z}) \to \\ = & & \cong & & \cong & & = \\ \to 0 & \longrightarrow & \bigoplus_{i=1}^{l_2} \mathbb{Z}Q \oplus \bigoplus_{j=1}^{k_2} \mathbb{Z}Q & \xrightarrow{\partial} & \bigoplus_{i=1}^{k_2} \mathbb{Z}Q & \longrightarrow & 0 \to \end{array}$$

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- Since $H_1(\overline{W_1}) = 0$, $\partial|_A$ is onto.
- Clearly, B is a free $\mathbb{Z}Q$-module.

Lemma

Let A, B, and C be R-modules, with B a free R-module (on the basis F), and let $\Theta : A \bigoplus B \to C$ be an R-module homomorphism. Suppose $\Theta|_A$ is onto. Then $\ker(\Theta) \cong \ker(\Theta|_A) \bigoplus B$.

- So, $\ker(\partial)$ is a free, finitely generated $\mathbb{Z}Q$-module

Proof of the Geometric Reverse

- So, $\ker(\partial)$ is a free, finitely generated $\mathbb{Z}Q$-module
- Also, $\pi_1(\overline{W_1})$ is superperfect.

Proof of the Geometric Reverse

Lemma (Superperfect Groups Have Spherical Elements for H_2)

Let P be a superperfect group. Let M be a manifold which has fundamental group isomorphic to P. Then any element of $H_2(M)$ can be killed by attaching 3-handles.

- This lemma may be seen as a direct corollary of the definition of superperfect.

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- Unfortunately, $\pi_1(W_1) \cong G$, not Q.

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- Finally, read W_2 from right to left.
- This creates a cobordism (W_3, N, M) with $[(n+1)-4]$-, $[(n+1)-3]$-, and $[(n+1)-2]$-handles added.

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- This creates a cobordism (W_3, N, M) with $[(n+1)-4]$-, $[(n+1)-3]$-, and $[(n+1)-2]$-handles added.
- This means $\pi_1(W_3) \cong Q$, as $n \geq 6$.

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- This concludes the proof

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- There actually is a finitely presented, superperfect, centerless, Hopfian group with torsion elements of infinitely many different orders
- Thompson's group V fits the bill
- It is fintely presented, superperfect, simple (hence centerless and Hopfian - also perfect), and contains a copy of each S_n, hence of each finite group, hence torsion elements of each order

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- For example, if $Q = \Pi_{i=1}^{\infty}\mathbb{Z}$, $K_1 = \mathbb{Z}$, and $K_2 = \mathbb{Z} \times \mathbb{Z}$, then $G_1 = K_1 \times Q$ and $G_2 = K_2 \times Q$ are isomorphic, even though $K_1 \ncong K_2$

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- This would open up more groups as possible candidates for P, for example, all fundamental groups of hyperbolic homology spheres of dimension ≥ 3

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Let $A, B, C,$ and D be nontrivial groups and ley $\phi : A \times B \to C * D$ be a epimorphism. Then either $\phi(A \times \{1\})$ is all of $C * D$ and $\phi(\{1\} \times B)$ is trivial or $\phi(A \times \{1\})$ is trivial and $\phi(\{1\} \times B)$ is all of $C * D$

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- Remark This is really the grain of sand that led to the pearl that is the theorem. Everything must commute and the domain, and nothing can commute in the range

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- The proof uses the fact that a free product is never an internal direct product

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Lemma (The Straightening-Up Lemma ($n = m$))

Let $n = m$, let S be a free product, and let $\psi : S \times S \times \ldots \times S$ (n copies) $\to S \times S \times \ldots \times S$ (m copies) be an isomorphism. Write $\psi_{i,j}$ for $\pi_{S_j} \circ \psi|_{S_i}$. Then ψ splits as n isomorphisms $\psi_{i,\sigma(i)}$, with σ a permutation, with all other $\psi_{i,j}$'s being the trivial map

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Let $n > m$, let S be a Hopfian free product, and let $\psi : S \times S \times \ldots \times S$ (n copies) $\to S \times S \times \ldots \times S$ (m copies) be an epimorphism. Write $\psi_{i,j}$ for $\pi_{S_j} \circ \psi|_{S_i}$. Then ψ splits as m isomorphisms $\psi_{\sigma^{-1}(i),i}$, with σ a permutation, with all other $\psi_{i,j}$'s being the trivial map

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- (This particular kind of outer action is called a partial conjugation)

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Let $n = m$, and let $\theta : G_1 \to G_2$ be an isomorphism. Then θ restricts to an isomorphism on the commutator subgroup $K = S \times S \times \ldots \times S$, and S factors which correspond by the Straightening-Up Lemma ($n = m$) have ϕ_u's with the same order in the definition of their semi-direct product

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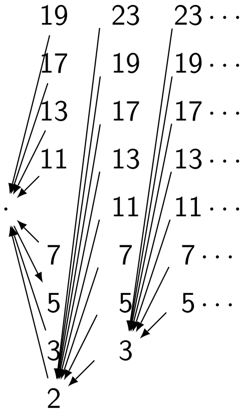

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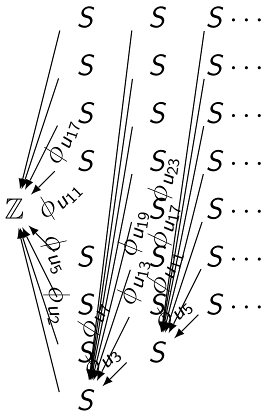

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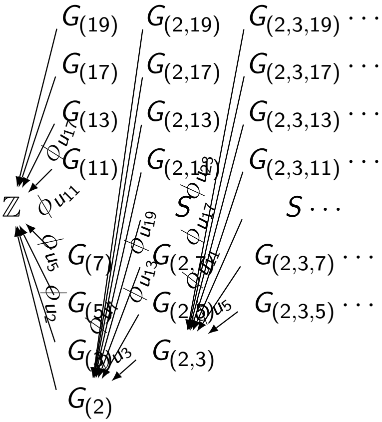

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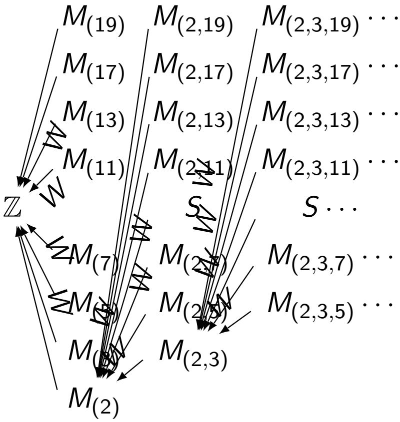

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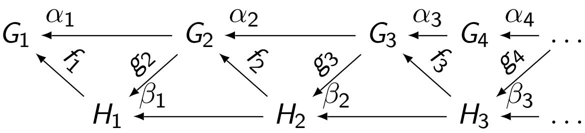
- Two inverse sequences are pro-isomorphic if and only if, after passing to subsequences, they may be put into a ladder diagram



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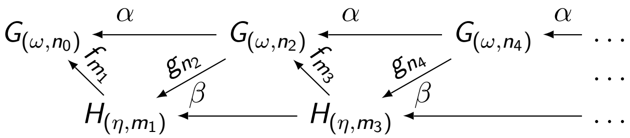

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- This concludes the proof

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- Setting $A_1 = A_0 \cup \{t_1\}$ and $R_1 = R_0 \cup \{t_1 = t_0 t_1^2 t_0\}$, we continue inductively setting $G_j = \langle A_{j-1}, t_j \mid R_{j-1}, t_j = t_{j-1} t_j^2 t_{j-1} \rangle$

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- By a theorem of Howie, as we can write $G_j = G_{j-1} *_{\langle t_j \rangle} I_2$, where I_2 is the Baumslag-Solitar group $\langle t, x \mid t = xt^2x \rangle$, G_j is hypo-Abelian.

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- Now, we produce the manifold V by starting with the manifold M_0 that has G_0 as its fundamental group, forming a cobordism (W, M_0, M_1), where M_1 has G_1 as its fundamental group, and proceeding inductively.

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- Now, we produce the manifold V by starting with the manifold M_0 that has G_0 as its fundamental group, forming a cobordism (W, M_0, M_1), where M_1 has G_1 as its fundamental group, and proceeding inductively.
- First, cross M_0 with $\mathbb{I}$, add a trivially attached 1-handle α_1^1 representing t_1 and a 2-handle α_2^2 representing $t_1 = t_0 t_1^2 t_0$, where t_0 is a loop of infinite order in M_0.

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- Now, $B_0 \cup \beta_1^2 \cup \beta_2^3 = B_0 \cup_{M_1} W_0$, where (W_0, M_1, M_0) is $M_1 \times \mathbb{I}$ with β_1^2 and β_2^3 attached

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- Read W_0 upside-down, so that it becomes (W_0, M_0, M_1), and W_0 looks like $M_0 \times \mathbb{I}$ with an $(n-3)$-handle γ_2^{n-3} and an $(n-2)$-handle γ_2^{n-2} attached

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- (W_0, M_0, M_1) is the cobordism we seek

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- Continue inductively forming (W_1, M_1, M_2), (W_2, M_2, M_3), *ad nfinitum*, and set $V = W_0 \cup_{M_1} W_1 \cup_{M_2} W_3 \ldots$

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- Then V has $G_0 \leftarrow G_1 \leftarrow G_2 \leftarrow \ldots$ as its fundamental group system at infinity and $\partial V = M_0$
- Finally, if we set $N_i = W_i \cup_{M_i} W_{i+1} \cup_{M_{i+1}} W_{i+2} \ldots$, $N_i' = \beta_i^2 \cup N_i$, and $M_i' = \beta_i^2 \cup M_i$, then $M_i' \hookrightarrow N_i'$ is a homotopy equivalence, so V is absolutely inward tame.

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- This completes the proof.

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