Some Constructions of Pseudo-Collarable 1-Ended Manifolds

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- In either case, the map f is a $\mathbb{Z}Q$-homology isomorphism

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- (Semi-direct products are the simplest kind of group extensions; direct products are one example)
- Then there is a cobordism (W, M, M_-) with $\pi_1(M_-) \cong G$ and $M \hookrightarrow W$ a simple homotopy equivalence

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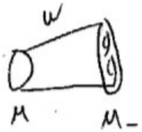

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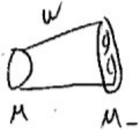

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- Note (W, M_-, M) (read upside-down, with the roles of M and M_- reversed) is a plus cobordism (so $(M_-)^+ \approx M$)

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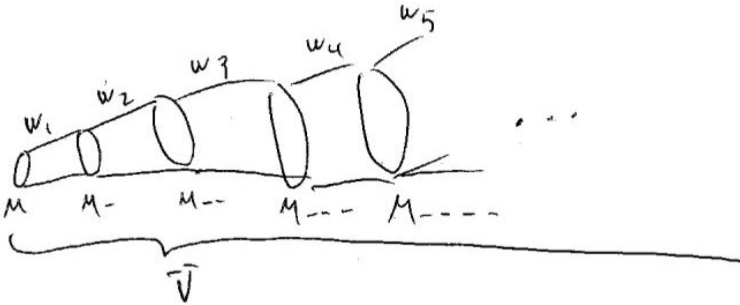
- 

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- This is one advantage of using semi-direct products over direct products

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Theorem (R., 2014)

Let M be a manifold of dimension $n \geq 6$ with fundamental group $\mathbb{Z}$. Let P be a finitely presented, superperfect, centerless, Hopfian group with torsion elements of infinitely many different orders, and let $S = P * P$. Then there are uncountably many $(n+1)$-dimensional, pseudo-collarable, 1-ended manifolds V with boundary M

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- It is fintely presented, superperfect, simple (hence centerless and Hopfian - also perfect), and contains a copy of each S_n, hence of each finite group, hence torsion elements of each order

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- The hard part is proving that the resulting pro-fundamental group systems at infinity are all non-isomorphic
- For example, if $Q = \Pi_{i=1}^{\infty}\mathbb{Z}$, $K_1 = \mathbb{Z}$, and $K_2 = \mathbb{Z} \times \mathbb{Z}$, then $G_1 = K_1 \times Q$ and $G_2 = K_2 \times Q$ are isomorphic, even though $K_1 \ncong K_2$

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- We would like to relax the constraint on P that it have torsion elements of infinitely many different orders to that it contains a countably infinite subset U with the property that there is no isomorphism ψ of P which carries u_i onto u_j for u_i and u_j distinct elements of U, but have run into some technical difficulties

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- This would open up more groups as possible candidates for P, for example, all fundamental groups of hyperbolic homology spheres of dimension ≥ 3

Sketch of Proof

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Lemma

Let A, B, C, and D be nontrivial groups and ley $\phi : A \times B \to C * D$ be a epimorphism. Then either $\phi(A \times \{1\})$ is all of $C * D$ and $\phi(\{1\} \times B)$ is trivial or $\phi(A \times \{1\})$ is trivial and $\phi(\{1\} \times B)$ is all of $C * D$

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- The proof uses the fact that a free product is never an internal direct product

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Lemma (The Straightening-Up Lemma ($n = m$))

Let $n = m$, let S be a free product, and let $\psi : S \times S \times \ldots \times S$ (n copies) $\to S \times S \times \ldots \times S$ (m copies) be an isomorphism. Write $\psi_{i,j}$ for $\pi_{S_j} \circ \psi|_{S_i}$. Then ψ splits as n isomorphisms $\psi_{i,\sigma(i)}$, with σ a permutation, with all other $\psi_{i,j}$'s being the trivial map

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Lemma (The Straightening-Up Corollary ($n > m$))

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- (This particular kind of outer action is called a partial conjugation)

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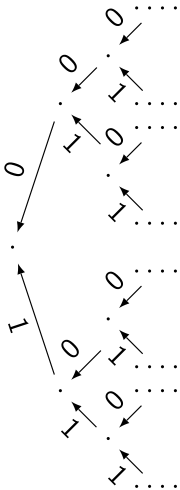

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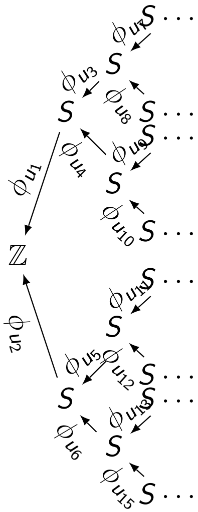

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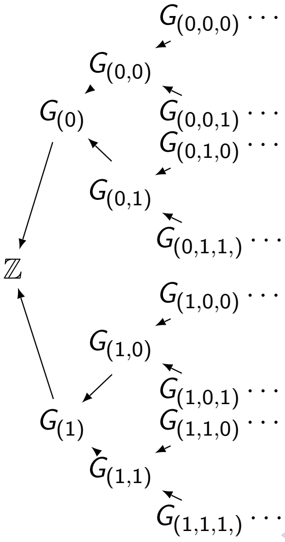

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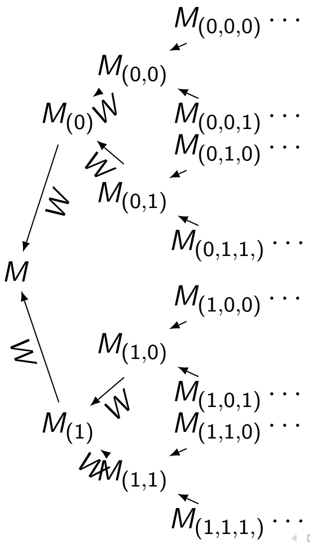

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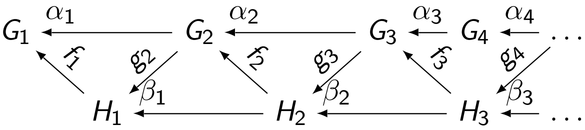
- Two inverse sequences are pro-isomorphic if and only if, after passing to subsequences, they may be put into a ladder diagram



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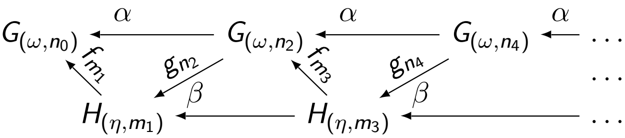
- Suppose, after passing to subsequences, we have their pro-fundamental group systems at infinity fitting into a ladder diagram



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- This concludes the proof

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