A Geometric Reverse to Quillen's Plus Construction

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- If the object is a CW cx, the Plus Construction simply create a map f between the two CW cxs.
- In either case, the map f is a $\mathbb{Z}Q$-homology isomorphism.

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- $GL(R)$ is the direct limit of all the $GL_n(R)$.
- BG is the classifying space of G.
- The perfect normal subgp by which we mod out is $E(\mathbb{Z}G)$, the subgroup generated by the elementary matrices.

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- Specifically, given a closed mfld N with fund gp $Q = \pi_1(N)$ and a f.p. superperfect group P, we construct a cpt one-sided h-cobordism W whose left-hand bdy component is N and whose right-hand boundary component is a closed mfld M whose fund. gp. $G = \pi_1(M)$ satisfies

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  and with $W \rightsquigarrow N$.
- Note that G and Q are necessarily f.p..

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- In 1976-8, J.C. Hausmann and P. Vogel investigated the problem of a reverse to the Plus Construction.
- Their results are are more comprehensive than ours.
- However, their work was unknown to us at the time we did our work.
- Also, although our work only creates one mfld for the reverse, instead of all possible mflds for reverses as Hausmann and Vogel do, it is possibly harder to determine when their results produce any reverses, and our work is somewhat more geometric than theirs.

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- A nbd of infinity U is an open collar nbd of infinity if and only if U is a mfld with cpt bdy and $U \approx \partial U \times [0,1)$.
- Siebenmann's Thesis gives necessary and sufficient conditions on an open n-mfld M that it admit an open collar nbd of infinity.

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- Open collar nbds of infinity are a special case of pseudo-collars; they are repetitions of trivial (product $\partial U \times \mathbb{I}$) reverse-plus construction cobordisms.
- Repeating reverse-plus constructions to produce pseudo-collars will be an important application of, and in fact was the motivation for, this work.

For further reference on the Group Extension Problem, the interested reader is referred to Homology by MacLane, pages 124-129.

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- Note that one solution to this problem is always just $Q \times K$. The classification gives all other solutions.
- This classification is given up to an isomorphism of two groups G and G' which makes a ladder of short exact sequences commute using the identity on K and Q, a so-called congruence of extensions.

$$\begin{array}{ccccccccc} 1 & \to & K & \to & G & \to & Q & \to & 1 \\ & & = & & \cong & & = & & \\ 1 & \to & K & \to & G' & \to & Q & \to & 1 \end{array}$$

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- An outer action of a gp Q on a gp K is a hom $\psi : Q \to Out(K)$.
- Given a gp G solving a given group extension problem, there is a canonical outer action of the quotient group Q on the kernel group K.
- So, to solve the Group Extension Problem, we begin with an outer action ψ of Q on K.

- An outer action ψ gives rise to an element $u(\psi) \in H^3(Q; Z(K))$, the third cohomology of Q with coefficients in the center of K.

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Solution to the Group Extension Problem

The congruence classes of solutions G to the Group Extension Problem with kernel K and quotient Q are in bijective correnspondence with $\mathcal{O}(Q, K) \times H^2(Q; Z(K))$.

Superperfect

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Definition

A gp P is superperfect if and only if $H_1(P) \cong H_2(P) \cong 0$.

- For the rest of this talk, all mflds are assumed to be orientable.

Theorem

Let $n \geq 6$. Let N be a closed n-manifold with $Q = \pi_1(N)$. Let P be f.p. and superperfect. Then there exists a cpt n-dimensional 1-sided h-cobordism (W, N, M) with left-hand bdy N and with right-hand bdy M having $\pi_1(M) \cong G$, where G is a gp extension of Q by P with trivial second homology elt and with $N \hookrightarrow W$ a simple homotopy equivalence.

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- However, we supply a new proof of this fact.

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- Take a small disk D inside of $N \times \{1\}$.
- Attach trivial 1-handles inside the disk for each generator β of P.
- Attach 2-handles across the disk for each relator s of P.

- As a corollary to the Solution to the Group Extension Problem, since we are using the trivial 2nd homology element, for each element $\beta_i \alpha_j$ in G, there is a word $w_{i,j}$ in the β's so that $\beta_i \alpha_j = \alpha_j w_{i,j}$.

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- Attach a 2-handle across each $\beta_i\alpha_j(\alpha_j w_{i,j})^{-1}$ (these will leak outside the disk).
- Now, we have a cobordism (W_1, N, M) with$\pi_1(N) \cong Q$, $\pi_1(W_1) \cong G$, and $\pi_1(M) \cong G$.

Lemma

Let $\overline{W_1}$ be the cover of W_1 with fund gp P (and covering transformation group Q). Then we may arrange the handles attached to $\overline{W_1}$ across $\overline{M}$ so that they project down equivariantly via the covering map to corresponding handles attached to W across M.

- So, now the relative handlebody chain complex of $(\overline{W_1}, \widetilde{N})$ looks like

$$\begin{array}{ccccccccccc}
\longrightarrow & C_3(\overline{W_1}, \widetilde{N}; \mathbb{Z}) & \longrightarrow & C_2(\overline{W_1}, \widetilde{N}; \mathbb{Z}) & \xrightarrow{\partial} & C_1(\overline{W_1}, \widetilde{N}; \mathbb{Z}) & \longrightarrow & C_0(\overline{W_1}, \widetilde{N}; \mathbb{Z}) & \longrightarrow \\
& = & & \cong & & \cong & & = & \\
\longrightarrow & 0 & \longrightarrow & \bigoplus_{i=1}^{l_2} \mathbb{Z}Q \oplus \bigoplus_{j=1}^{k_2} \mathbb{Z}Q & \xrightarrow{\partial} & \bigoplus_{i=1}^{k_2} \mathbb{Z}Q & \longrightarrow & 0 & \longrightarrow
\end{array}$$

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- Since $H_1(\overline{W_1}) = 0$, $\partial|_A$ is onto.

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- Since $H_1(\overline{W_1}) = 0$, $\partial|_A$ is onto.
- Clearly, B is a free $\mathbb{Z}Q$-module.

Lemma

Let A, B, and C be R-modules, with B a free R-module (on the basis F), and let $\Theta : A \bigoplus B \to C$ be an R-module homomorphism. Suppose $\Theta|_A$ is onto. Then $\ker(\Theta) \cong \ker(\Theta|_A) \bigoplus B$.

- So, $\ker(\partial)$ is a free, finitely generated $\mathbb{Z}Q$-module

- So, $\ker(\partial)$ is a free, finitely generated $\mathbb{Z}Q$-module
- Also, $\pi_1(\overline{W_1})$ is superperfect.

Lemma (Superperfect Groups Have Spherical Elements for H_2)

Let P be a superperfect group. Let M be a manifold which has fundamental group isomorphic to P. Then any element of $H_2(M)$ can be killed by attaching 3-handles.

- This lemma may be seen as a direct corollary of the definition of superperfect.

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- Unfortunately, $\pi_1(W_1) \cong G$, not Q.

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- Finally, read W_2 from right to left.
- This creates a cobordism (W_3, N, M) with $[(n+1)-4]$-, $[(n+1)-3]$-, and $[(n+1)-2]$-handles added.

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- This creates a cobordism (W_3, N, M) with $[(n+1)-4]$-, $[(n+1)-3]$-, and $[(n+1)-2]$-handles added.
- This means $\pi_1(W_3) \cong Q$, as $n \geq 6$.

- Now, $\pi_1(\widetilde{W_3}) \cong 1$.

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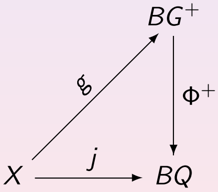
- This proves $W_3 \rightsquigarrow N$
- Because of the special nature of the group extension G and by an otherwise well-known technique (Rourke & Sanderson, P. 90), we may adjust the torsion on M so that $N \hookrightarrow W_3$ is a simple homotopy equivalence.

- THE END

Definition

$\{X, BG^+\}$ is defined to be the subset of $[X, BG^+]$ represented by $g : X \to BG^+$ such that
(1) $g_\# : \pi_1(X) \to Q$ makes the following diagram commute on fundamental groups:



(2) $\pi_2(g) : \pi_2(X) \to \pi_2(BG^+)$ is onto

Corollary (Hausmann and Vogel, 1978)

Let M be a closed CAT-manifold of dimension $n \geq 5$ with fundamental group Q. Let X be the associated CW complex of cores and let $q : M \to X$ be the simple homotopy equivalence between M and X. Let P be a locally perfect group and let G be a group extension of Q by P. Then the set of all semi-s-cobordisms (W, M, M_-) with left-hand boundary M and right-hand boundary having fundamental group isomorphic to G is in bijective correspondence with $\{X, BG^+\}$.

Hausmann and Vogel's Results

Theorem (Hausmann, 1978)

Let M be a closed CAT-manifold of dimension $n \geq 5$ with $\pi_1(M) = Q$. Let $\Phi : G \twoheadrightarrow Q$ be a homomorphism with kernel P, where P is f.p., locally perfect (every f.g. subgroup is perfect), and superperfect. Assume one of the following conditions is realized:
(a) $\Phi^+ : BG^+ \rightarrow BQ$ admits a homotopy section $s : BQ \rightarrow BG^+$ such that the following diagram is homotopy commutative:

$$\begin{array}{ccccc} BG & & \xrightarrow{\quad\iota\quad} & & BG^+ \\ \uparrow{\scriptstyle\mu} & & & & \uparrow{\scriptstyle s} \\ \partial M & \xrightarrow{\iota} & M & \xrightarrow{j} & BQ \end{array}$$

(Con'd)

Theorem (Con'd)

(b) $H^i(BQ, \partial M; \pi_{i-1}(BG^+)) = 0$ *for* $4 \leq i \leq n$
(c) $H^i(M, \partial M; \pi_{i-1}(BG^+)) = 0$ *for* $i \geq 4$
Then there exists a compact CAT-manifold M_- *with* $\partial(M_-) = \partial M$ *and a map* $f : M_- \to M$ *such that*
(1) $\pi_1(M_-) \cong G$ *and* $f_\# = \Phi$
(2) $\omega_1(M_-) = f^*(\omega_1(M))$*, where* ω_1 *is the first Stiefel-Whitney class*
(3) f_* *is a* $\mathbb{Z}Q$*-homology isomorphism.*